ISSN 2229-5518

Unitary combinations of formalized classes in qubit space

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Abstract - The formalization of qubit operators has two main objectives. The first one is to present all qubit operators as linear combinations of Ext_1 and $N\mathrm{ext}_1$, i.e. identity and negation. Both classes are analogous to the classic operators and therefore the formalization of single qubit operators with these classes, acting as main operators, provides means for operation with primitive operators, set in the classical concepts for calculations. The second objective is to be separated the parts of the phase of the state to be separated from those of the amplitude in such a way, so as to be set out the key models at the disturbance, generated by the operators.

Index Terms— Boolean function, circuit, composition, encoding, gate, phase, quantum.

1 INTRODUCTION

Both classes of Ext_1 and $N\mathrm{ext}_1$, i.e. identity and negation are analogous to the classic operators and therefore the formalization of single qubit operators with these classes, acting as main operators, provides means for operation with primitive operators, set out in the classical concepts for calculations. The main objective is the parts of the phase of the state to be separated from those of the amplitude in such a way so as to be set out the key models at the disturbance, generated by the operators. This disturbance is a key to the quantum calculations and the development of quantum algorithms. In addition, the separation allows for the strict characterization of the consequences form a change of the phase as the encoding of binary information in a phase space and in this way allows the characterization in terms of Boolean functions. In this report will be discussed the necessary and sufficient conditions for combining operators from Ext_1 and $N\mathrm{ext}_1$ for formation of unitary operators and provision of an abstraction on the relative weights of the base operators. The logical formalization of the main operators will be addressed in respect of their phases and phase changes, which they carry out, on the states, to which they are applied.

2 Formalization of qubit operators
2.1 Unitary combinations of Id and Neg

If first is examined the space of the single qubit operators, since they serve as elementary constructive elements for the operators over many qubits. Equation 1 lists the full set from single qubit operators.

$$\mathbf{Id_{00}} = |\mathbf{0}\rangle\langle\mathbf{0}| + |\mathbf{1}\rangle\langle\mathbf{1}| = \boldsymbol{I}$$

$$\mathbf{Id_{01}} = |\mathbf{0}\rangle\langle\mathbf{0}| - |\mathbf{1}\rangle\langle\mathbf{1}| = \boldsymbol{Z}$$

$$\mathbf{Id_{10}} = -|\mathbf{0}\rangle\langle\mathbf{0}| + |\mathbf{1}\rangle\langle\mathbf{1}| = -\boldsymbol{Z}$$

$$\mathbf{Id_{11}} = -|\mathbf{0}\rangle\langle\mathbf{0}| - |\mathbf{1}\rangle\langle\mathbf{1}| = -\boldsymbol{I}$$

$$\mathbf{Neg_{00}} = |\mathbf{0}\rangle\langle\mathbf{1}| + |\mathbf{1}\rangle\langle\mathbf{0}| = \boldsymbol{X}$$

$$\mathbf{Neg_{01}} = |\mathbf{0}\rangle\langle\mathbf{1}| - |\mathbf{1}\rangle\langle\mathbf{0}| = -\boldsymbol{N}$$

$$\mathbf{Neg_{10}} = -|\mathbf{0}\rangle\langle\mathbf{1}| + |\mathbf{1}\rangle\langle\mathbf{0}| = \boldsymbol{N}$$

$$\mathbf{Neg_{11}} = -|\mathbf{0}\rangle\langle\mathbf{1}| - |\mathbf{1}\rangle\langle\mathbf{0}| = -\boldsymbol{X} \qquad \mathbf{(1)}$$

As shown on equation 1, the space of the operators for identity and negation for single qubits is generated through the function *Id* and *Neg*. If a basic set {*I*, *Z*, *N*, *X*} is given, then it is possible to be determined which combination forms unitary operators. Of particular interest are the requirements for these phased parameters. When one set from phased parameters comes from {00, 11}, the other must be from {01, 10}.

Formal prerequisite 1 *If* $\boldsymbol{U = aId_x + bNeg_y}$ is an operator of two dimensional Hilbert space above the real numbers. For 0 < *a*, *b* < *1*, *U* is unitary only if $\boldsymbol{a^2+b^2=1}, \boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$ and $\boldsymbol{x} \in \{\mathbf{01},\mathbf{10}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$.

Proof. It is assumed that *U is* unitary, then, $\boldsymbol{U^\dagger U = I}$ and therefore, $\boldsymbol{U_{00}^2 + U_{01}^2 = U_{10}^2 + U_{11}^2 = 1}$ and $\boldsymbol{U_{00}U_{10} + U_{01}U_{11} = U_{10}U_{00} + U_{11}U_{01} = 0}$. When

$$\boldsymbol{U_{00}} = \pm\boldsymbol{a}$$

$$\boldsymbol{U_{01}} = \pm\boldsymbol{b}$$

$$\boldsymbol{U_{10}} = \pm\boldsymbol{b}$$

$$\boldsymbol{U_{11}} = \pm\boldsymbol{a}$$

follows that $\boldsymbol{a^2+b^2=b^2+a^2=1}$. If it is accepted that $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\}$, then $\boldsymbol{a(U_{10}+U_{01})}$ can be zero only when $\boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$. Similarly, if $\boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$, then $\boldsymbol{b(U_{00}-U_{11}) = 0}$ only when $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\}$. If $\boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$, then, $\boldsymbol{b(U_{00}+U_{11})}$ can be zero only for $\boldsymbol{x} \in \{\mathbf{01},\mathbf{10}\}$ and when $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\}$, then $\boldsymbol{aU_{01}-aU_{10}}$ is zero only at $\boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$. If it is accepted that $\boldsymbol{a^2+b^2=1}$ and $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$, and $\boldsymbol{x} \in \{\mathbf{01},\mathbf{10}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$. If $\boldsymbol{a^2+b^2=1}$, then $\boldsymbol{(UU^\dagger)_{00} = (UU^\dagger)_{11} = 1}$. In order *U* to be unitary, must be shown that $\boldsymbol{(UU^\dagger)_{01} = (UU^\dagger)_{10} = 0}$, when $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$ and $\boldsymbol{x} \in \{\mathbf{01},\mathbf{10}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$. At $\boldsymbol{x} \in \{\mathbf{00},\mathbf{11}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{01},\mathbf{10}\}$ $\boldsymbol{U_{00} = U_{11} = a}$ and $\boldsymbol{U_{01} = b = -U_{10}}$. From this follows that $\boldsymbol{(UU^\dagger)_{01} = (UU^\dagger)_{10} = 0}$. Similarly, when $\boldsymbol{x} \in \{\mathbf{01},\mathbf{10}\} \leftrightarrow \boldsymbol{y} \in \{\mathbf{00},\mathbf{11}\}$, then $\boldsymbol{U_{00}} = -\mathrm{a} = \boldsymbol{-U_{11}}$ and $\boldsymbol{U_{01} = U_{10} = b}$ and once

ISSN 2229-5518

more $(\boldsymbol{UU^\dagger})_{\mathbf{01}} = (\boldsymbol{UU^\dagger})_{\mathbf{10}} = \mathbf{0}$. Formal prerequisite 1 determines the necessary and sufficient conditions for combining operators from $\mathrm{Ex}\boldsymbol{t_1}$ and $\boldsymbol{N}\mathrm{ex}\boldsymbol{t_1}$ for formation of unitary operators. As it appears, can be combined only elements of $\boldsymbol{Int_1}$ with $\boldsymbol{Nint_1}$ and of $\mathrm{Ex}\boldsymbol{t_1}\backslash\boldsymbol{Int_1}$ with $\boldsymbol{N}\mathrm{ex}\boldsymbol{t_1}\backslash \boldsymbol{Nint_1}$. This observation leads to a logical formalization of single qubit operators in contrast to the more structural formalization, given in Formal prerequisite 1.

Abstraction of amplitudes

From Formal prerequisite 1 it is clear that for unitary operator $\boldsymbol{U = aId_x + bNeg_y}$ the amplitudes *a* and *b* must comply with the restriction $\boldsymbol{a^2 + b^2 = 1}$. Since the functions *Neg* and *Id* effectively control the characters of each record in *U*, it is sufficient to be examined only positive values for *a and b*. This, in its turn, allows to be used the values of the amplitudes $\boldsymbol{a'}$ and $\boldsymbol{b'}$ such that $\boldsymbol{a' + b' = 1}$. Therefore, for each unitary operator with real values $\boldsymbol{U = aId_x + bNeg_y}$, satisfying the conditions, given in Formal prerequisite 1 can be expressed the amplitudes *a* and *b* through a single parameter $\boldsymbol{\alpha}$ such that

$$\boldsymbol{U = \sqrt{a}Id_x + \sqrt{1-a}Neg_y} \tag{2}$$

Logical formalization of a phase space

The space of the single qubit, main operators with real values can be logically formalized by three binary axes:

1. The parameter of *the global phase* $\boldsymbol{\gamma}$ determines whether a given gate is a combination of $\pm\boldsymbol{I}$ and $\pm\boldsymbol{N}$ or $\pm\boldsymbol{Z}$ and $\pm\boldsymbol{X}$.
2. The parameter of *the phase of identity* $\boldsymbol{\iota}$ determines the phase of the operator for identity.
3. The parameter of *the phase of negation* $\boldsymbol{\eta}$ determines the phase of the operator for negation.

Both global phases separate the logical operators into intensional and purely extensional subsets. Both $\boldsymbol{I}$, and $\boldsymbol{N}$ - the foundations of one of the global phases - are intensional operators, at which $\boldsymbol{Z}$ and $\boldsymbol{X}$ are extensional and not intensional. Therefore, the combinations of $\boldsymbol{I}$ and $\boldsymbol{N}$ may be taken as *partial intensional operators*. Similarly the combinations of $\boldsymbol{Z}$ and $\boldsymbol{X}$ can be considered strictly as *partial extensional operators*. The parameter $\boldsymbol{\gamma}$ selects the extent to which a given operator acts by an intensional / extensional manner. Functions will be defined, which connect the logical phase parameters with the more structural formalization of *Id* and *Neg*. For the achievement of this objective let's $\boldsymbol{\gamma = 0}$ be the combination of the intensional operators, and $\boldsymbol{\iota = 0}$ and $\boldsymbol{\eta = 0}$ indicate the positive phase respectively for the operators for identity and negation. This leads to the next enumeration, based on the $\boldsymbol{\gamma\iota\eta}$.

$\mathbf{000} \to \boldsymbol{I + N}$ $\mathbf{100} \to \boldsymbol{Z + X}$

$\mathbf{001} \to \boldsymbol{I - N}$ $\mathbf{101} \to \boldsymbol{Z - X}$

$\mathbf{010} \to \boldsymbol{-I + N}$ $\mathbf{110} \to \boldsymbol{-Z + X}$

$\mathbf{011} \to \boldsymbol{-I - N}$ $\mathbf{111} \to \boldsymbol{-Z - X}$

With this allocation of space it is possible to be defined functions, which connect the enumeration based on the logical parameters with the parameters of the function *Id* and *Neg*.

Definition 1 *The parameters* $(\boldsymbol{\gamma}, \boldsymbol{\iota})$ *are connected with* $(\boldsymbol{\iota}, \boldsymbol{\gamma \oplus \iota})$ *through the function* $\boldsymbol{\mathfrak{T}_{\gamma\eta} = Id_{i(\gamma \oplus \iota)}}$*, defined as*

$$\boldsymbol{\mathfrak{T}_{00} = Id_{00} = I}$$

$$\boldsymbol{\mathfrak{T}_{01} = Id_{11} = -I}$$

$$\boldsymbol{\mathfrak{T}_{10} = Id_{01} = Z}$$

$$\boldsymbol{\mathfrak{T}_{11} = Id_{10} = -Z} \tag{3}$$

Definition 2 *The parameters* $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ *are connected with* $((\overline{\boldsymbol{\gamma \oplus \eta}}), \boldsymbol{\eta})$ *through the function* $\boldsymbol{\mathcal{N}_{\gamma\eta} = Neg_{(\overline{\gamma \oplus \eta})\eta}}$*, defined as*

$$\boldsymbol{\mathcal{N}_{00} = Neg_{10} = N}$$

$$\boldsymbol{\mathcal{N}_{01} = Neg_{01} = -N}$$

$$\boldsymbol{\mathcal{N}_{10} = Neg_{00} = X}$$

$$\boldsymbol{\mathcal{N}_{11} = Neg_{11} = -X} \tag{4}$$

Both functions provide an opportunity for defining the desired connection from the phase tripled $\boldsymbol{\gamma\iota\eta} \in \mathbb{B}^3$ with pairs of the phase doubled $\mathbb{B}^2 \mathbf{x}\, \mathbb{B}^2$, corresponding respectively to the arguments *Id* and *Neg*.

$$\boldsymbol{\gamma\iota\eta} = \left(\boldsymbol{\mathfrak{T}_{\gamma\iota}}, \boldsymbol{Neg_{\gamma\eta}}\right) \tag{5}$$

2.2 Final formalization

This chapter began with the objective for formalization of the space of the single qubit operators in a way that is motivated from the extensional and intensional base operators. Such formalization is already possible. If $\mathcal{U}_1$ is the space of unitary operators of the two-dimensional Hilbert space above $\mathbb{R}$ and $\mathbb{R}_{[0,1]}$ is the interval [0,1]. It is defined $U : \mathbb{R}_{[0,1]} \times \mathbb{B}^3 \to \mathcal{U}_1$ through:

$$U(\alpha, \gamma\iota\eta) = \sqrt{\alpha}\mathfrak{T}_{\gamma\iota} + \sqrt{1-\alpha}\mathcal{N}_{\gamma\eta} \tag{6}$$

Now it can be shown that U fully describes $\mathcal{U}_1$.

Formal prerequisite 2 *(Presentation of* U_1*) For each* $A \in \mathcal{U}_1$ *exist* $0 \le \alpha \le 1$ *and* $\gamma\iota\eta \in \mathbb{B}^3$ *such that* $U(\alpha, \gamma\iota\eta) = A$*.*

Proof. If $\alpha = A_{0,0}^2$. Since $A_{0,0}^2 + A_{0,1}^2 = 1$, then $A_{0,1} = \pm\sqrt{1-\alpha}$. A is orthogonal, $AA^\dagger = I$, and $A_{0,0} = \pm\sqrt{\alpha}$, $A_{1,1} = \pm\sqrt{\alpha}$ and $A_{1,0} = \pm\sqrt{1-\alpha}$. For each combination of these values, A can be recorded as $\sqrt{\alpha}\mathfrak{T}_{\gamma\iota} + \sqrt{1-\alpha}\mathcal{N}_{\gamma\eta}$

When the parameter of the amplitude α is in the interval (0, 1), the operators are uniquely defined from their parameters. If $\alpha = 0$ or $\alpha = 1$, which corresponds to the main operators, there are two-to-one connections between parameters and operators.

In order to be avoided any ambiguity, all basic virtualizations will accept a parameter with a zero value for the phase of the inactive basis. This means that, at $\alpha = 1$, then $\eta = 0$, respectively at $\alpha = 0$ $\iota = 0$. Table 1 shows the correct formalization of

ISSN 2229-5518

the main operators.

Table 1: Formalization of the main operators

		$\gamma = 0$		$\gamma = 1$	
		$\eta = 0$	$\eta = 1$	$\eta = 0$	$\eta = 1$
$\alpha = 1$	$\iota = 0$	I		Z	
	$\iota = 1$	-I		-Z	
$\alpha = 0$	$\iota = 0$	N	-N	X	-X
	$\iota = 1$				

Formalization of the properties of a matrix

Some relations between matrices are important for the work with the operators, which they represent. Now these relations will be established with respect to the parameters of the operators.

Formal prerequisite 3 *If* $A = U(\alpha, \gamma\iota\eta)$.

$$A^{\dagger} = A^{T} = \begin{cases} A & \gamma = 1 \\ U(\alpha, \gamma\iota\bar{\eta}) & \gamma = 0 \end{cases} \tag{7}$$

Proof. Since all the formalized operators are orthogonal matrices, then it is true that $A^{\dagger} = A^{T}$. At $\gamma = 1$ the phase of the diagonal elements A_{01} and A_{10} is always one and the same and therefore $A = A^{\dagger}$. At $\gamma = 0$ the diagonal elements A_{01} and A_{10} always differ in their phase. Transposition of the matrix must swap the phase of negation of the operator. From 1.3.2 follows that, for $U(\alpha, \gamma\eta)$ at $\gamma = 1$ the operator is Hermitean.

Formal prerequisite 4 *If* $A = U(\alpha, \gamma\iota\eta)$. *Then* $-A = U(\alpha, \gamma, \bar{\iota}, \bar{\eta})$.

Proof. Follows from the multiplication of each record of the matrix for *A* with -1.

Algebra of the formalized operators

The formalization of the operators, developed in this system, provides algebra for the composition of the two operators. Table 2 shows the Cayley table for composition of main operators with positive phase. This composition corresponds to the multiplication of the matrix presentation of each operator.

Table 2: Cayley table for single qubit main operators

∘	I	N	Z	X
I	I	N	Z	X
N	N	- I	X	- Z
Z	Z	- X	I	-N
X	X	Z	N	I

The operators for extensional identity have very well defined structure, as shown in formal prerequisite 5.

Formal prerequisite 5 *The set* Ext_1 *with the operator for composition* ° *is one group.*

Proof. Ext_1 is closed under the operator for composition, as shown in table 1.2. The Operator *I* is the identity of the composition. The associativity of Ext_1 below ° follows from the associativity of standard multiplication of matrices. For each $A \in \mathrm{Ext}_1$ exists inverse $A^{-1} \in \mathrm{Ext}_1$. As can be expected, the combination of any two extensional operators for negation gives as a result an extensional operator for identity. In a very broad sense, this keeps the classic double negation. In addition, the combination of an operator for identity with an operator for negation leads to negation.

Formal prerequisite 6 *If the operators A and in come from the set of the main operators* $\mathrm{Ext}_1 \cup Next_1$. *Then*

$$BA \in \begin{cases} \mathrm{Ext}_1 & A, B \in \mathrm{Ext}_1 \text{ или } A, B \in Next_1 \\ Next_1 & A \in \mathrm{Ext}_1, B \in Next_1 \text{ или } A \in Next_1, B \in \mathrm{Ext}_1 \end{cases} \tag{8}$$

Proof. The proof follows from the the Cayley table in table 5.2 and formal prerequisite 1.3.4.

Formal prerequisite 7 *For a, b, c, d* ∈ {0, 1}

$$\mathfrak{I}_{ab} \circ \mathfrak{I}_{ab} = \mathfrak{I}_{(a \oplus c)(b \oplus d)}$$
$$\mathcal{N}_{ab} \circ \mathcal{N}_{cd} = \mathfrak{I}_{(a \oplus c)(\bar{a} \oplus b \oplus d)}$$
$$\mathfrak{I}_{ab} \circ \mathcal{N}_{cd} = \mathcal{N}_{(a \oplus c)(a \oplus b \oplus d)}$$
$$\mathcal{N}_{cd} \circ \mathfrak{I}_{ab} = \mathcal{N}_{(a \oplus c)(b \oplus d)} \tag{9}$$

Proof. These equations can be checked with the help of the the Cayley table.

The composition of random formalized operators with main operators can be expressed in terms of the transformations of parameters.

Formal prerequisite 8 *If the operator* $A = U(\alpha, \gamma\iota\eta)$. *Then*

$$\mathfrak{I}_{00}A = A\mathfrak{I}_{00} = A$$
$$\mathfrak{I}_{01}A = A\mathfrak{I}_{01} = -A$$
$$\mathcal{N}_{00}A = U\big((1-\alpha), \gamma\iota\bar{\eta}\big)$$
$$\mathfrak{I}_{10}A = U(\alpha, \bar{\gamma}\iota\bar{\eta})$$
$$\mathcal{N}_{10}A = U\big((1-\alpha), \bar{\gamma}\eta\iota\big)$$
$$A\mathfrak{I}_{10} = U(\alpha, \bar{\gamma}\eta\iota)$$
$$A\mathcal{N}_{10} = U\big((1-\alpha), \gamma(\bar{\gamma} \oplus \eta)(\gamma \oplus \iota)\big) \tag{10}$$

Proof. If *A* is a base operator. Then

$$A \circ U(a, \gamma\iota\eta) = A \circ \left(\sqrt{a}\mathfrak{I}_{\gamma\eta} + \sqrt{1-a}\mathcal{N}_{\gamma\eta}\right)$$
$$= \sqrt{a}A \circ \mathfrak{I}_{\gamma\eta} + \sqrt{1-a}A \circ \mathcal{N}_{\gamma\eta}$$

The change, or the absence of such, of the parameter of the amplitude follows from formal prerequisite 6, because at $A \in Next_1$ the main operators will effectively turn against the parameter of the amplitude. The transformations of the phase follow from the Cayley table 1.2. As it is shown in Formal prerequisite 10, the operator for composition is not commutative. When an operator is combined with a negative basis with another operator formal prerequisite 4 can be applied together with Formal prerequisite 9, in order to be obtained a correct result from the composition of an operator with a formalized operator.

Examples

If at first is examined an operator, which can be called square root of *N*, i.e. $\sqrt{N}$. The operators for a square root are classic example for the difference between the probability amplitudes and probabilities, and the influence of the disturbances. The operator $\sqrt{N}$ is formalized through $U(\frac{1}{2}, 000)$, and, therefore is a combination of *I* and *N*, as shown in Example 1

Example 1

ISSN 2229-5518

$$\sqrt{N} = U(\frac{1}{2}, 000)$$

$$= \sqrt{\frac{1}{2}}\mathfrak{T}_{00} + \sqrt{\frac{1}{2}}\mathcal{N}_{00}$$

$$= \sqrt{\frac{1}{2}}Id_{00} + \sqrt{\frac{1}{2}}Neg_{01}$$

$$= \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

3 CONCLUSION

The formalized classes separate the parts for phase and amplitude and allow for expression of the amplitude in terms of probabilities for difference from the more general probabilistic amplitudes. The necessary and sufficient conditions for construction of unitary, in this case orthogonal, operators are integrated in the formalization itself. Either directly, or with the aid of extensions this formalizations acts as a basis for a fully functional system for encoding and decoding of operators. Formal prerequisite 8 and the supporting formal prerequisite 6 and formal prerequisite 7 form an important part of the overall abstraction, since they allow operation with formalized operators without necessarily resorting to the main matrix representation of the operator.